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CYLINDRICITY OF ISOMETRIC IMMERSIONS BETWEEN HYPERBOLIC SPACES

BY

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ABSTRACT. The motivation for this paper was to prove the following analogue of the Euclidean cylinder theorem: any umbilic-free isometric immersion $\eta: H^{n-1} \to H^n$ between hyperbolic spaces takes the form of a hyperbolic $(n-2)$-cylinder over a uniquely determined parallelizing curve in $\bar{H}^n$. Our approach is through the more general study of isometric immersions generated by one-parameter families of hyperbolic k-planes without focal points. A by-product of this study is a natural extension to curves in $\bar{H}^n$ of the notion of a parallel family of k-planes along a curve in H^n; the extension is based on spherical symmetry of variation fields. Existence and uniqueness properties of this extended notion of parallelism are considered.

By a well-known theorem of Hartman and Nirenberg, every isometric immersion $\eta: E^{n-1} \to E^n$ between Euclidean spaces takes the form of a Euclidean $(n-2)$-cylinder [2]. That is, $\eta(E^{n-1})$ may be obtained by parallel translation of an $(n-2)$-plane in E^n along a base curve, where infinitely many choices of base curve are clearly possible. The analogous problem of determining the isometric immersions $\eta: H^{n-1} \to H^n$ between hyperbolic spaces of curvature -1 was posed by Nomizu in [4]. One may modify this problem by considering only the immersions which are umbilic-free, that is, have nowhere vanishing second fundamental form. Such an immersion is well known to be generated by *hyperbolic* $(n-2)$-*planes* (complete, $(n-2)$-dimensional, totally geodesic submanifolds) in H^n [5]. The umbilic-free isometric immersions of H^{n-1} into H^n were characterized by Ferus in [1]. Ferus' analysis is concerned with the orthogonal trajectories of the generators (for example, see Theorem 3 below), and does not touch on the question of parallelization.

This paper studies isometric immersions $\eta: M \to H^n$ generated by one-parameter families, without focal points, of hyperbolic k-planes. For such an immersion, the sectional curvatures of M satisfy $K_M \leq -1$; if $K_M = -1$ then M is isometric to an open subset of H^{k+1}. The original motivation for this study was to prove Theorem 6, which states that any umbilic-free isometric immersion of H^{n-1} into H^n takes the form of a hyperbolic $(n-2)$-cylinder

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over a uniquely determined parallelizing curve in $\bar{H}^n$ (see further remarks below). Here $\bar{H}^n = H^n \cup H^n_\infty$, where H^n_∞ denotes the equivalence classes of asymptotic geodesic rays in H^n. This theorem offers an interesting analogue to the Euclidean cylinder theorem. The possibility of such an analogue was suggested by the classification of developable surfaces in H^3, Theorem 2, [6].

In §1, for any immersion η: $M \to H^n$ generated by a family of hyperbolic k-planes, the existence of a uniquely determined *striction curve* is obtained (Theorem 1). The striction point on a given k-plane is defined in terms of the *variation field* of the family; this is a vector field defined on and orthogonal to the k-plane, and "tangent to the family". If furthermore $K_M = -1$, it is shown that the striction curve is also the unique parallelizing curve in $\bar{H}^n$ of the family, so that the immersion takes the form of a hyperbolic k-cylinder (Theorem 2).

Here it is necessary to extend to curves in $\bar{H}^n$, the obvious notion of a *parallel* family of hyperbolic k-planes along a curve in H^n. This is done in a natural way, by proving first that a family $\{H_c\}_{c \in I}$ of k-planes is parallel along a regular curve σ: $I \to H^n$ if and only if for each c, the variation field is covariantly constant on the metric spheres in H_c about $\sigma(c)$ (Proposition 1). Then the extension to curves σ: $I \to \bar{H}^n$ is obtained simply by substituting "horospheres" for "metric spheres" if $\sigma(c) \in H^n_\infty$.

§2 considers foliations of H^n by $(n-1)$-planes, and contains a criterion for generating such a foliation by parallel translation along a curve in H^n (Theorem 4). §3 studies isometric immersions of H^{k+1} into H^n generated by families of k-planes. For example, it is shown that any parallel family $\{H_c\}$ along a unit-speed curve σ: $R \to H^n$, for which the angle between the family and σ is bounded away from zero, generates an immersion of H^{k+1} into H^n (Theorem 5). Umbilic-free isometric immersions of H^{n-1} into H^n are also discussed in §3.

§4 examines the implications of the extension to curves in $\bar{H}^n$ of the notion of parallel k-plane families. A differentiable manifold-with-boundary structure is placed on $\bar{H}^n$, namely that induced by the projective model of H^n. Along a regular curve σ lying on H^n_∞, the parallel translation problem has a unique solution for any allowable initial condition (Theorem 8). Of course, unique solutions always exist when σ lies in H^n. Solutions do not generally exist if σ takes both finite and infinite values (the question concerns piecing together solutions of two different problems, because of constraints which operate at H^n_∞). However, if a solution exists, it is unique (Corollary 4).

The method of §4 is to convert the study of hyperbolic k-plane families into the study of Euclidean k-plane families, by using the projective model of H^n. Instead of the parallelizing curve of a hyperbolic family, one may examine the family of focal sets of a Euclidean family (Theorem 7). The classical approach to focal families (that is, to characteristics and envelopes) seems to be highly computational. Our approach is concise, coordinate-free, and new as far as we know.

Finally, in illustration of the preceding section, §5 examines families of hyperbolic k-planes having parallelizing curves in $\bar{H}^n$ of certain special types.

Our conventions are as follows. On H^n, g or $\langle\ ,\ \rangle$ denotes the Riemannian metric, ∇ denotes covariant differentiation, and D denotes distance. *Geodesics* are always parametrized by arc-length. In §§1–3, no differentiable structure is assumed for $\bar{H}^n$. A *curve* is simply a map of an interval unless otherwise described. "Corollary 1.1" refers to part 1 of Corollary 1.

1. Families of k-planes in H^n. Let $(T_1, \ldots, T_k)$ be a smooth map of an open interval I into the bundle of k-frames on H^n, and H_c be the hyperbolic k-plane tangent to $(T_1, \ldots, T_k)(c)$ for each $c \in I$. Then the parametrized k-planes $\{H_c\}_{c\in I}$ will be called a (*one-parameter*) *family* in H^n. The map $(T_1, \ldots, T_k)$ will be called a *frame* for the family, and its projection $\alpha = \pi \circ (T_1, \ldots, T_k)\colon I \to H^n$, a *base curve*.

Such a family of hyperbolic k-planes determines a *variation vector field* V_c on each of its members H_c, defined as follows: V_{cp} is the component of $\beta'(c)$ orthogonal to H_c, for any smooth curve β in H^n satisfying $\beta(c) = p$ and $\beta(c + \delta) \in H_{c+\delta}$, $|\delta| < \varepsilon$. To see that the orthogonal components for all such curves β agree, one may observe that if any one such β is transverse to H_c at p, then the $H_{c+\delta}$ locally determine a $(k + 1)$-dimensional submanifold $M(p)$ through p in H^n, and are the level hypersurfaces of a regular function x on $M(p)$. Since by assumption every β satisfies $x \cdot \beta = \mathrm{id}$, then with respect to the H^n-induced metric on $M(p)$, for every β the component of $\beta'(c)$ orthogonal to the level hypersurface H_c is $\|\mathrm{grad}\, x\|^{-2}$ grad x.

The zeroes of the variation field V_c form the *focal set* at c of the family $\{H_c\}$. The family is *regular* at c if V_c does not vanish identically.

Given a frame for the family, define $\eta\colon I \times \mathbf{R}^k \to H^n$ by

$$\eta(c, c_1, \ldots, c_k) = \exp \sum_{i=1}^{k} c_i T_i(c). \tag{1}$$

Denote by x, x_i and X, X_i respectively the coordinate functions and vector fields on $I \times \mathbf{R}^k$. Then $\eta_* X$ at $x = c$ is an H^n-Jacobi field on H_c (that is, Jacobi along every geodesic in H_c). Since by definition, V_c coincides with the component of $\eta_* X$ orthogonal to H_c, it follows from the form of Jacobi fields in H^n that V_c is Jacobi. Therefore the length function

$$l_c = \|V_c\| \tag{2}$$

on H_c (carrying the H^n-induced metric) is convex, and strictly convex where nonvanishing. Since H_c is isometric to H^k, the following lemma applies:

LEMMA 1. *For any convex function $f\colon H^k \to \mathbf{R}$, the limit of f along a geodesic ray in a fixed asymptote class is independent of the choice of ray.*

PROOF. By convexity, $f \circ \gamma$ has a limit in $[-\infty, \infty]$ along any geodesic ray γ. For $j = 1, 2$, suppose $\lim f \circ \gamma_j = n_j$, where $n_1 < n_2$ and γ_1 and γ_2 are distinct asymptotic rays. Choose sequences $\{p_{1i}\}$ and $\{p_{2i}\}$ diverging to infinity along γ_1 and γ_2 respectively, and a point q in the hyperbolic 2-plane containing γ_1 and γ_2, such that p_{2i} lies on the geodesic segment δ_i joining p_{1i} and q. Then $D(p_{1i}, p_{2i})$ is bounded while $D(p_{2i}, q)$ becomes arbitrarily large. Since $f \circ \delta_i$ is convex, $n_1 < n_2$ would contradict finiteness of $f(q)$. Therefore $n_1 = n_2$, as required.

Now let the *variation function* $\bar{l}_c\colon \bar{H}_c \to [0, \infty]$ be the extension, whose existence is guaranteed by Lemma 1, of the length function given by (2).

LEMMA 2. *If a family $\{H_c\}$ of k-planes in H^n has no focal points at c, then the variation function at c has a single minimum point, which is the limit as $\delta \to 0$ of the point of $\bar{H}_c$ closest to $H_{c+\delta}$.*

PROOF. It is a consequence of strict convexity that l_c has at most one critical point and $\bar{l}_c$ has exactly one minimum point. (If $\bar{l}_c$ is finite at a point $p \in H_{c\infty}$, then l_c must decrease on all geodesic rays to p.)

For $p \in H_c$, let $\beta_p(c + \delta)$ be the point of $H_{c+\delta}$ nearest p. Then it is not hard to see that $\beta_p = \exp_p kN$ for smooth function k and vector-valued function N, where $k(c) = 0$ and $N(c) \perp H_c$. Since $\beta_p'(c) = k'(c)N(c)$, which is orthogonal to H_c, then $V_{cp} = \beta_p'(c)$. It follows that

$$l_c(p) = \lim_{\delta \to 0^+} \left[D(p, H_{c+\delta})/\delta \right]. \tag{3}$$

The function $D(\cdot, H_{c+\delta})$ on H_c is strictly convex, and therefore extends to $\bar{H}_c$ with exactly one minimum point p_δ. From (3) and the fact that convergence of convex functions on Riemannian manifolds is uniform on compact subsets, we may conclude easily that p_δ converges to the minimum point of $\bar{l}_c$, as was claimed. This completes the proof of Lemma 2.

Next suppose $\{H_c\}_{c \in I}$ is a family *without focal points*. Let $M = \{(c, p)\colon c \in I, p \in H_c\}$, and define $\eta\colon M \to H^n$ by $\eta(c, p) = p$. A frame $(T_1, \ldots, T_k)$ for the family determines an obvious coordinatization of M by $I \times R^k$, for which η is expressed by (1); and thus determines a differentiable structure on M, with respect to which η is an immersion. This structure is independent of choice of frame. Letting M carry the η-induced Riemannian metric, we say that the isometric immersion $\eta\colon M \to H^n$ is *generated* by the family, and has *generators* H_c. In what follows, we shall sometimes suppress η, identifying M locally with its image in H^n.

Observe that by the Gauss equation, sectional curvatures of M satisfy $K_M \leqslant -1$, with equality holding if and only if the normalized variation field

$l_c^{-1}V_c$ is covariantly constant on H_c for all $c \in I$.

For each $c \in I$ the variation function $\bar{l}_c$ has a unique minimum point, which by Lemma 2 it is reasonable to call the *striction point*. Note that the variation function on a given k-plane depends only up to a constant multiple on the choice of regular parametrization of the family (equivalently, of the base curve α), so the striction point is independent of that choice. If $\bar{l}_c(p) = 0$ for some $p \in H_{c\infty}$, we say the family is *asymptotic* at p; this terminology is justified by (3).

THEOREM 1. *Let η: $M \to H^n$ be the isometric immersion generated by a family $\{H_c\}_{c \in I}$ of hyperbolic k-planes without focal points. Then there is a uniquely determined curve σ: $I \to \overline{H}^n$ which satisfies*: (i) *$\sigma(c) \in \overline{H}_c$ for all $c \in I$*; (ii) *the generators are M-parallel along $\sigma|J$ when $\sigma|J$ lies in H^n*; (iii) *the generators are asymptotic at $\sigma(c)$ when $\sigma(c) \in H^n_\infty$. The curve σ coincides with the curve of striction points.*

PROOF. Note that here we regard $\overline{H}_c$ as lying in $\overline{H}^n$, in the natural way. Given $c_0 \in I$, it suffices to prove that the theorem holds for some open subinterval about c_0. Therefore there is no loss of generality in assuming that M is coordinatized with respect to a base curve α *orthogonal* to the H_c. Thus we assume M carries coordinate vector fields $X, X_1, \ldots, X_k$ for which (supressing η and η_*) the X_i are tangent to the H_c; the restriction of X to each H_c is Jacobi in H^n; along α, $X = \alpha'$ and the X_i are covariantly constant in the normal bundle of α in M. Let γ be a geodesic in H_c satisfying $\gamma(0) = \alpha(c)$. By parallelism of X_i, $\nabla_{\alpha'(c)}(\Sigma c_i X_i)$ is orthogonal to H_c, where $\gamma'(0) = \Sigma c_i X_{i\alpha(c)}$ and ∇ is covariant differentiation in H^n. Therefore $\nabla_{\gamma'(0)}X$ is orthogonal to H_c. Since X is H^n-Jacobi along γ, X is orthogonal to H_c along γ. Therefore by the definition of variation field,

$$X|H_c = V_c \tag{4}$$

and at any point of H_c,

$$\langle \nabla_X X_i, X \rangle = \langle \nabla_{X_i} X, X \rangle = (X_i l_c) l_c. \tag{5}$$

Now let $\sigma(c)$ denote the striction point at c. If $\sigma|J$ lies in H^n then $\sigma|J$ is smooth (this follows from the implicit function theorem and the fact that $\sigma(c)$ is a nondegenerate critical point of l_c). Since $\{H_c\}$ has no focal points, $\sigma|J$ is regular. By (5), since $\sigma(c)$ is a critical point of l_c and H_c is totally geodesic,

$$\langle \nabla_{\sigma'} X_i, X \rangle = \langle \nabla_X X_i, X \rangle = 0.$$

Therefore the tangent k-planes to the H_c are M-parallel along $\sigma|J$, as required. Conversely, if σ^*: $J^* \to H^n$ is any curve in H^n satisfying $\sigma^*(c + \delta) \in H_{c+\delta}$ and parallelizing the H_c in M, then $\sigma^*(c)$ is critical for l_c, so $\sigma^*(c) = \sigma(c)$ for all $c \in J^*$.

By the Jacobi equation, if $U_1, \ldots, U_{n-k}$ are covariantly constant orthonormal fields defined on and orthogonal to H_c, then

$$V_c = \sum_{j=1}^{n-k} h_j U_j \tag{6}$$

where the h_j are *hyperbolic* functions on H_c, that is, where $(h_j \circ \gamma)'' = h_j \circ \gamma$ for every geodesic γ in H_c. If $\gamma(0) = q$, $V_{cq} = \Sigma a_j U_{jq}$ and $\nabla_{\gamma'(0)} V_c = \Sigma b_j U_{jq}$, then

$$h_j \circ \gamma(s) = a_j \cosh s + b_j \sinh s, \qquad 1 < j < n - k. \tag{7}$$

If the striction point $\sigma(c)$ lies at H^n_∞ and γ is a geodesic ray to $\sigma(c)$, the limit of $h_j \circ \gamma$ is finite for all j, by definition of $\sigma(c)$. But then $a_j = -b_j$ in (7), and the limit of h_j along γ vanishes for all j. Therefore $\bar{l}_c(\sigma(c)) = 0$. Conversely, if $\bar{l}_c(p) = 0$ for some $p \in H_{c\infty}$, obviously p is the striction point at c. This completes the proof of Theorem 1.

The proof of Theorem 1 has the following corollary, of which the second part is essentially due to Ferus [1]:

COROLLARY 1. *Let* η: $M \to H^n$ *be the isometric immersion generated by a family* $\{H_c\}$ *of hyperbolic k-planes without focal points.* 1. *The curvature vector in M of an orthogonal trajectory to the generators at any* $q \in H_c$ *is* $(-l_c^{-1} \operatorname{grad} l_c)_q$.

2. *If M has constant sectional curvature* -1, *then the curvature in M of such a curve does not exceed* 1.

PROOF. 1. Our notation will be that of the preceding proof. Since X is orthogonal to the H_c, the curvature vector in question is the orthogonal projection tangent to H_c of $\|X\|^{-2}\nabla_X X$ at q. But by (5) and (4),

$$\langle \|X\|^{-2}\nabla_X X, X_i \rangle = -l_c^{-1} X_i l_c = \langle -l_c^{-1} \operatorname{grad} l_c, X_i \rangle.$$

2. If M has constant sectional curvature -1, then in (6) we may take $h_1 = l_c$ and $h_j = 0$ for $j > 1$. Thus $l_c \circ \gamma(s) = a \cosh s + b \sinh s$ for any geodesic γ in H_c, by (7), where $|a^{-1}b| \leqslant 1$ since $l_c > 0$. If γ is chosen so that

$$\gamma(0) = q \quad \text{and} \quad \gamma'(0) = \left(-\|\operatorname{grad} l_c\|^{-1} \operatorname{grad} l_c\right)_q,$$

then $a = l_c(q)$, $b = -\|\operatorname{grad} l_c\|(q)$, and $|a^{-1}b|$ is the required curvature at q by part 1.

REMARK 1. By Corollary 1.1, an orthogonal trajectory is an M-geodesic if and only if it coincides with the striction curve.

PROPOSITION 1. *A family* $\{H_c\}$ *of hyperbolic k-planes is* H^n*-parallel along an arbitrary regular base curve* σ *in* H^n *if and only if for each c, the variation field at c is covariantly constant on metric spheres in* H_c *about* $\sigma(c)$.

PROOF. If V_c is covariantly constant on metric spheres about some $q \in H_c$, then for h_j as in (6), $(h_j \circ \gamma)'(0) = 0$ for any geodesic γ in H_c with $\gamma(0) = q$; hence $b_j = 0$ for all j, in (7). It follows that V_c is covariantly constant on metric spheres in H_c about q if and only if

$$V_{cp} = a \cosh D(p, q) U_p \tag{8}$$

where U is a covariantly constant unit normal to H_c.

Now suppose $V_{c\sigma(c)} \neq 0$. Locally the family generates an imbedded submanifold which, as in the proof of Theorem 1, carries coordinate vector fields $X, X_1, \ldots, X_k$. Here the restriction of X to each member of the family coincides with the variation field, the X_i are tangent to the family, and (5) is satisfied. If V_c is covariantly constant on spheres about $\sigma(c)$, then $\sigma(c)$ is critical for the length function l_c. Furthermore, on H_c, $\nabla_{X_i}X$ is proportional to X by (8). Therefore $\nabla_X X_i = 0$ at $\sigma(c)$ by (5), so that $\nabla_{\sigma'(c)}X_i$ is tangent to H_c, as required. Conversely, if $\nabla_{\sigma'(c)}X_i$ is tangent to H_c, then so is $\nabla_X X_i$ at $\sigma(c)$, since $\sigma'(c)$ is transverse to H_c by assumption. Since $\langle \nabla_{X_i}X, X_j\rangle = -\langle \nabla_{X_i}X_j, X\rangle = 0$, it follows that $\nabla_{X_i}X = 0$ at $\sigma(c)$. Therefore by (7), V_c is covariantly constant on spheres about $\sigma(c)$.

Suppose instead that $V_{c\sigma(c)} = 0$. Define a map η: $I \times \mathbf{R}^k \to H^n$ by (1), for any choice of frame over σ. It suffices to show that V_c is covariantly constant on spheres about $\sigma(c)$ if and only if if the k vectors

$$Y_i = \nabla_{\sigma'(c)}\big(\eta_* X_i|_{x^1 = \cdots = x^k = 0}\big) = \nabla_{\eta_* X_i}\big(\eta_* X|_{x=c}\big) \quad \text{at } \sigma(c)$$

are tangent to H_c. If V_c is constant on spheres about $\sigma(c)$, then V_c vanishes identically, by (8). Since V_c is the component of $\eta_* X|_{x=c}$ orthogonal to H_c, it follows that the Y_i are tangent to H_c. Conversely, if the Y_i are tangent to H_c, then for any geodesic γ with $\gamma(0) = \sigma(c)$, we have $b_j = 0$ in (7). Since $a_j = 0$ by assumption, V_c vanishes identically. Thus Proposition 1 is proved.

We shall call a family $\{H_c\}_{c\in I}$ of hyperbolic k-planes *parallel* in $\overline{H}^n$ along a curve σ: $I \to \overline{H}^n$ if for each c, $\sigma(c) \in \overline{H}_c$, and the variation field at c is covariantly constant on metric spheres about $\sigma(c)$ if $\sigma(c) \in H_c$, or on horospheres about $\sigma(c)$ if $\sigma(c) \in H_{c\infty}$. (In the latter case, the variation function will be shown to vanish at $\sigma(c)$.) By Proposition 1, this definition is simply an extension of the usual one.

PROPOSITION 2. *Let η: $M \to H^n$ be the isometric immersion generated by a family $\{H_c\}_{c\in I}$ without focal points, and let σ: $I \to \overline{H}^n$ be the striction curve of the family. Then M has constant sectional curvature -1 if and only if the family is parallel in $\overline{H}^n$ along σ.*

PROOF. Recall that $K_M = -1$ if and only if for each c, $l_c^{-1}V_c$ is covariantly constant. We claim that this occurs if and only if V_c is covariantly constant on spheres about $\sigma(c)$. Indeed, if $\sigma(c) \in H_c$ and V_c is constant on metric spheres about $\sigma(c)$, then $l_c^{-1}V_c$ is constant by (8). If $\sigma(c) \in H_\infty^n$, then $l_c^{-1}V_c$ is constant on each geodesic ray to $\sigma(c)$ by the remarks following (7); thus constancy on horospheres about $\sigma(c)$ implies constancy on all of H_c.

For the converse, consider any H^n-Jacobi field on H_c of the form lU, where $l > 0$, U is a given covariantly constant unit normal to H_c, and l has a given value at some fixed point of H_c. Such a field is determined by choice of minimum point $q \in \overline{H}_c$ for l, since l is hyperbolic. Since l is a constant multiple of $\cosh D(\cdot, q)$ when q is finite, then l is constant on metric spheres about q when $q \in H_c$, and by a limit argument, on horospheres about q when $q \in H_{c\infty}$.

THEOREM 2. *A regular family $\{H_c\}_{c\in I}$ of hyperbolic k-planes is without focal points and generates an immersed submanifold of constant curvature* -1 *if and only if the family is parallel in $\overline{H}^n$ along some curve σ: $I \to \overline{H}^n$. If such a parallelizing curve σ exists, it is unique.*

PROOF. It suffices, by Proposition 2, to show that if the family is regular and parallel in $\overline{H}^n$ along some σ, then there are no focal points and σ is the striction curve of the family. But if V_c is covariantly constant on metric spheres about some $q \in H_c$, then V_c is given by (8). Thus V_c never vanishes because it does not vanish identically, and q is the striction point at c. If V_c is constant on horospheres about $q \in H_{c\infty}$, then again V_c never vanishes, because otherwise the zero set of V_c would be a hyperbolic r-plane for $0 \leq r < k$, and no such r-plane contains a horosphere of H_c. Furthermore, since the striction point is the *unique* minimum point of the length function $\bar{l}_c$, this point is certainly q.

REMARK 2. A submanifold of the type considered in Theorem 2 is always isometric to an open subset of H^{k+1} (see Proposition 3).

REMARK 3. We shall see in §4 that if $\overline{H}^n$ is given the manifold-with-boundary structure induced by the projective model of H^n, then the parallelizing curve σ of Theorem 2 is smooth in $\overline{H}^n$. Furthermore, when σ lies at infinity there are constraints on the derivatives of σ. For example, if $k = 1$, σ' vanishes at infinity (Corollary 2); thus if σ lies at infinity on an interval, it is constant there, and the immersion takes the form of an *asymptotic cone.*

2. **Foliations of H^n by hyperplanes.** Let $\mathfrak{F}$ denote a foliation of H^n by hyperbolic $(n-1)$-planes. We shall require the following important theorem of Ferus concerning such foliations:

THEOREM 3 [1]. 1. *For any $p \in H^n$ and any regular curve α in H^n with curvature $\kappa \leq 1$, the function $D(\alpha(t), p)$ has at most one critical point, namely a strict relative minimum point, and is unbounded if α has infinite length. In consequence*:

2. *The family of orthogonal hyperplanes to a regular curve with* $\kappa < 1$ *is without self-intersections.*

3. *For a regular curve with* $\kappa \leqslant 1$ *and unbounded length in both directions, the family of orthogonal hyperplanes foliates* H^n.

REMARK 4. A family $\{H_c\}$ of hyperplanes in H^n is without focal points if and only if it is without self-intersections. Corollary 1.2 and Theorem 3.2 imply that if there are no focal points then there are no self-intersections. On the other hand, since the variation field for an arbitrary hyperplane family, possibly with focal points, is of the form hU where U is covariantly constant and h is hyperbolic, the focal set in H_c is nonempty only if it is an $(n-2)$-plane on either side of which h takes opposite signs. Observing that h is given by the expression in (3), where D is interpreted as *directed* distance toward U, one may show that each focal point is the limit of intersection points of H_c and $H_{c+\delta}$ as $\delta \to 0$. Thus our remark is verified.

Given a foliation $\mathfrak{F}$ of H^n by hyperplanes, one always may choose a regular curve $\alpha: I \to H^n$ of infinite length which is everywhere orthogonal to $\mathfrak{F}$. By Corollary 1.2 and Theorem 3.3, the family of hyperplanes orthogonal to α is all of $\mathfrak{F}$. Therefore $\mathfrak{F}$ corresponds to a regular one-parameter family $\{H_c\}_{c\in I}$ (without focal points) of hyperplanes in H^n, and by Theorem 2, possesses a uniquely determined parallelizing curve $\sigma: I \to \overline{H}^n$.

The following theorem gives a necessary and sufficient condition that a given curve σ in H^n and initial transverse hyperplane determine by parallel translation a foliation of H^n.

THEOREM 4. *Let* $\sigma: \mathbf{R} \to H^n$ *be a unit-speed curve,* H_0 *a hyperbolic* $(n-1)$-*plane through* $\sigma(0)$, *and* H_c *the parallel translate of* H_0 *along* σ *to* $\sigma(c)$.

1. $\{H_c\}$ *is without focal points if and only if* H_c *is transverse to* $\sigma'(c)$ *for all* $c \in \mathbf{R}$.

2. *Assuming that* $\{H_c\}$ *is without focal points, let* $\theta(c) \in (0, \pi/2]$ *be the angle between* H_c *and* $\sigma'(c)$. *Fix* $q \in H^n$. *Then* $\{H_c\}$ *foliates* H^n *if and only if*

$$\int_0^\infty f(x)\,dx = \infty \quad \textit{and} \quad \int_{-\infty}^0 f(x)\,dx = \infty,$$

where $f(x) = \sin\theta(x)\cosh D(q, \sigma(x))$. *In particular*:

3. $\{H_c\}$ *foliates* H^n *if* θ *is bounded away from zero.*

PROOF. 1. By (8) and the definition of variation field, $V_{cp} = a \cdot \cosh D(p, \sigma(c))\, U_p$, where U is a covariantly constant unit normal to H_c and $a = \langle \sigma'(c), U\rangle = \sin\theta(c)$. Thus V_c is nowhere vanishing if $\theta(c) \neq 0$ and vanishes identically if $\theta(c) = 0$.

2. Let $\alpha: I \to H^n$ be a maximal curve satisfying $\alpha(c) \in H_c$ and $\alpha'(c) = V_{c\alpha(c)}$ for all $c \in I$. Since $\{H_c\}_{c\in I}$ is without focal points and α is an

orthogonal trajectory, the curvature of α does not exceed 1. The length of α is unbounded in both directions if and, by Theorem 3.3, only if $I = \mathbf{R}$ and $\{H_c\}$ foliates H^n.

Suppose $\{H_c\}$ does not foliate H^n. We may assume α has finite length in the positive direction. Finiteness and maximality of α imply that the domain of α includes $[d, \infty)$ for some d. Then

$$\begin{aligned}\|\alpha'(x)\| &= \sin\theta(x)\cosh D(\alpha(x), \sigma(x))\\ &\geqslant \sin\theta(x)\cosh\left[D(q, \sigma(x)) - D(q, \alpha(x))\right]\\ &\geqslant L f(x)\end{aligned}$$

for some $L > 0$ independent of $x \in [d, \infty)$, since $D(q, \alpha(x))$ is bounded on $[d, \infty)$. Therefore $\int_d^\infty f(x)\,dx < \infty$.

Now, suppose $\{H_c\}$ foliates H^n. Choose α to pass through q. By Theorem 3.1, $D(\alpha, \sigma(x)) = D(\alpha(x), \sigma(x)) \leqslant D(q, \sigma(x))$ for each x. Therefore $\|\alpha'(x)\| \leqslant f(x)$, and since α has unbounded length in both directions, the required divergence of integrals follows.

EXAMPLE 1. Let $\sigma\colon \mathbf{R} \to H^n$ be any unit-speed curve with total curvature

$$\int \|\nabla_{\sigma'}\sigma'\| < \pi,$$

with each of

$$\int_0^\infty \|\nabla_{\sigma'}\sigma'\| \quad \text{and} \quad \int_{-\infty}^0 \|\nabla_{\sigma'}\sigma'\|$$

less than $\pi/2$; and let H_0 be orthogonal to σ. Then the family of parallel translates foliates H^n.

Indeed, suppose I is a maximal interval about 0 on which the family is transverse to σ, and let U be the unit normal to the family along $\sigma|I$ in the direction of $\sigma'(c)$. Then

$$|\langle\sigma', U\rangle'| = |\langle\nabla_{\sigma'}\sigma', U\rangle| \leqslant \|\nabla_{\sigma'}\sigma'\|\cos\theta,$$

whence $|\theta'| \leqslant \|\nabla_{\sigma'}\sigma'\|$, where θ is the angle between σ' and H_c. Therefore our hypothesis implies that θ is bounded away from 0 on I. But then $I = \mathbf{R}$ and the claim follows by Theorem 4.3.

3. Isometric immersions of H^{k+1} into H^n. First we take note of the global structure of an immersed submanifold of H^n of the type considered in Theorem 2:

PROPOSITION 3. *Let $\eta\colon M \to H^n$ be the isometric immersion generated by a family of hyperbolic k-planes without focal points. If $K_M = -1$ then M is isometric to an open submanifold of H^{k+1}.*

PROOF. Since M and H^{k+1} are analytic and locally isometric, and since M

is simply connected, there exists an isometric immersion $\iota: M \to H^{k+1}$ ([3, pp. 255 and 263]). By assumption, M is foliated by a family $\{\tilde{H}_c\}$ of complete, totally geodesic submanifolds (where $\eta(\tilde{H}_c) = H_c$). But then $\iota(\tilde{H}_c)$ is a k-plane in H^{k+1} for each c. Since ι is an immersion, the family $\{\iota(\tilde{H}_c)\}$ is without focal points. By Remark 3, $\{\iota(H_c)\}$ is without self-intersections, and it follows that ι is one-one.

REMARK 5. Consider an isometric immersion $\eta: M \to H^n$ such as is described in Proposition 3, and identify M with an open submanifold of H^{k+1}. Then M is foliated by a family $\{\tilde{H}_c\}$ of hyperbolic hyperplanes, where $\eta(\tilde{H}_c) = H_c$. Denote by $\tilde{\sigma}: I \to \overline{H}^{k+1}$ and $\sigma: I \to \overline{H}^n$ the parallelizing curves of the families $\{\tilde{H}_c\}$ and $\{H_c\}$ respectively (the existence of $\tilde{\sigma}$ and σ is guaranteed by Theorem 2), and by $\eta_c: \overline{(\tilde{H}_c)} \to \overline{H}_c$ the natural identification extending $\eta|\tilde{H}_c$. If $\tilde{V}_c$ and V_c are the variation fields at c for the respective families, then $\eta_* \tilde{V}_c$ is orthogonal to H_c since η is isometric. Therefore $\eta_* \tilde{V}_c = V_c$, and $\eta_c \circ \tilde{\sigma}(c) = \sigma(c)$ for all $c \in I$.

The following construction of a class of isometric immersions of H^{k+1} into H^n depends upon Theorem 4:

THEOREM 5. *Let $\sigma: \mathbf{R} \to H^n$ be a unit-speed curve, H_0 a hyperbolic k-plane through $\sigma(0)$, and H_c the parallel translate of H_0 along σ to $\sigma(c)$. Let $\theta(c)$ be the angle between $\sigma'(c)$ and H_c, for each $c \in \mathbf{R}$. If θ is bounded away from zero, then the family $\{H_c\}$ generates an isometric immersion $\eta: H^{k+1} \to H^n$. The immersion η is umbilic-free if and only if the curvature vector in H^n of σ at c is transverse to the span of $\sigma'(c)$ and H_c for each $c \in \mathbf{R}$.*

PROOF. The family $\{H_c\}$ is regular since it is transverse to σ. Therefore by Theorem 2, $\{H_c\}$ generates an isometric immersion $\eta: M \to H^n$ where M has curvature -1. By Proposition 3, M may be identified with an open submanifold of H^{k+1}. Since η is an isometric immersion, we have a unit-speed curve $\tilde{\sigma}: \mathbf{R} \to M$, and a family $\{\tilde{H}_c\}$ of hyperbolic k-planes which foliates M and is parallel along $\tilde{\sigma}$, and whose angle with $\tilde{\sigma}$ is bounded away from zero (where $\eta \circ \tilde{\sigma} = \sigma$ and $\eta(\tilde{H}_c) = H_c$). But then by Theorem 4.3, $\{\tilde{H}_c\}$ foliates H^{k+1}, and so $M = H^{k+1}$.

Let K be the open submanifold of H^{k+1} on which the nullity of the second fundamental form of η takes its minimum value ν. Since all tangent vectors to the $\tilde{H}_c$ are nullity vectors, $k \leq \nu \leq k+1$. Recall that the nullity foliation of K is well known to have complete leaves: for the present situation, namely immersion of H^{k+1} into H^n, completeness was proved in [5]. The transversality hypothesis on the curvature vectors of σ implies that nullity on K is k and $\tilde{\sigma}$ lies in K. Therefore K contains the complete k-plane $\tilde{H}_c$ for every $c \in \mathbf{R}$, and so $K = H^{k+1}$ and η has no umbilics. Conversely, if there are no umbilics, then $\nu = k$ along $\tilde{\sigma}$, and the transversality condition follows. This completes the proof of Theorem 5.

Finally we consider umbilic-free isometric immersions $\eta: H^{n-1} \to H^n$. For such an immersion, H^{n-1} is foliated by hyperbolic $(n-2)$-planes tangent to the nullity spaces of the second fundamental form. As in the previous section, the foliating $(n-2)$-planes may be regarded as a regular one-parameter family $\{\tilde{H}_c\}_{c\in I}$ in H^{n-1}. Since η is totally geodesic on each $\tilde{H}_c$, η is generated by a family $\{H_c\}_{c\in I}$, without focal points, of $(n-2)$-planes in H^n. Then Theorem 2 immediately yields:

THEOREM 6. *Let $\eta: H^{n-1} \to H^n$ be an umbilic-free isometric immersion and $\{H_c\}_{c\in I}$ be the family of generators of η. Then there is a uniquely determined curve $\sigma: I \to \overline{H}^n$ along which the generators are parallel in $\overline{H}^n$.*

4. **Parallelization at H^n_∞.** In order to study the behavior at infinity of the parallelizing curve of a family, we shall give $\overline{H}^n$ the differentiable manifold-with-boundary structure of $\overline{D}^n$ induced by the projective model. By working with the projective model, we shift our attention to families of Euclidean k-planes in E^n and their intersection properties. Although the methods of this section are Euclidean, note that all the results except Theorem 7 are results about $\overline{H}^n$ carrying the given differentiable structure.

g^* or $\cdot$ will denote the standard Riemannian metric on E^n, and D^* will denote Euclidean distance. Any family $\{H_c^*\}$ of Euclidean k-planes in E^n has a variation field V_c^* at c, which is defined just as before but in terms of g^*, and which is a g^*-Jacobi field on H_c^*. Thus

$$V_c^* = \sum_{j=1}^{n-k} h_j^* U_j^* \tag{9}$$

where the U_j^* are constant orthonormal fields defined on and orthogonal to H_c^*, and the h_j^* are affine functions on H_c^*. The focal set F_c of $\{H_c^*\}$ at c is the intersection of the zero sets of the h_j^*, and if nonempty is an r-plane (possibly "at infinity") for some $r \geqslant 2k - n$. (Since the solution set of any homogeneous linear equation on a projective space is a projective hyperplane, we are justified in adopting the convention that any affine function on a Euclidean space vanishes on a hyperplane, which lies "at infinity" if the function is constant.) Since two affine functions are constant multiples of each other if and only if they have the same zeros, F_c is a $(k-1)$-plane if and only if we may assume $h_j^* = 0$ for all $j > 1$ in (9).

For the rest of this section, H^n will be identified with the Euclidean open unit disc D^n carrying the projective hyperbolic metric g. Unparametrized H^n-geodesics are thus identified with line segments in D^n, and totally geodesic submanifolds of H^n with totally geodesic submanifolds of D^n.

Asymptotic geodesics in H^n correspond to segments approaching the same point of $\partial D^n = S^{n-1}$. Thus H^n_∞ is identified with S^{n-1}, and $\bar{H}^n$ with $\bar{D}^n$.

To each k-plane of a given family $\{H_c\}$ in H^n may be associated the complete Euclidean k-plane H_c^* satisfying $H_c = H_c^* \cap D^n$. The hyperbolic family $\{H_c\}$ is without focal points if and only if for each c, the focal set of $\{H_c^*\}$ does not enter D^n. (Note that focal points depend only on differentiable structure, not on metric structure.) Such a family $\{H_c\}$ generates an immersed submanifold of H^n. The g-induced metric on this submanifold has curvature -1 if and only if for each c, the hyperbolic variation field V_c is tangent to a fixed hyperbolic $(k+1)$-plane through H_c; hence if and only if the Euclidean variation field of $\{H_c^*\}$ is tangent to a fixed Euclidean $(k+1)$-plane through H_c^*. Equivalent conditions on $\{H_c^*\}$ are that the metric induced by g^* on the submanifold generated by $\{H_c^* \cap D^n\}$ is flat; or (by the second paragraph of this section) that the focal sets F_c of $\{H_c^*\}$ are all $(k-1)$-planes.

By Theorem 2, a hyperbolic family $\{H_c\}$ generating an immersion of curvature -1 has a unique parallelizing curve in $\bar{H}^n$. The following theorem relates the parallelizing curve of $\{H_c\}$ in $\bar{H}^n$ to the focal family of $\{H_c^*\}$ in E^n via the classical notion of duality of points and hyperplanes. For any $p \in \bar{D}^n$, let p^* denote the inversion of p through S^{n-1}; then the hyperplane through p^* and orthogonal to the line Opp^* is the *dual* to p through S^{n-1}.

THEOREM 7. *Let $\{H_c\}_{c \in I}$ be a regular family of hyperbolic k-planes which is parallel in $\bar{H}^n$ along a curve $\sigma: I \to \bar{H}^n$, and let $\{H_c^*\}$ be the associated Euclidean family. Then the focal set F_c of $\{H_c^*\}$ is the $(k-1)$-plane dual to $\sigma(c)$ through $H_c^* \cap S^{n-1}$.*

PROOF. By Theorem 2 and remarks above, the hyperbolic and Euclidean families $\{H_c\}$ and $\{H_c^*\}$ have variation fields lU and l^*U^* respectively at c, where U is a covariantly constant unit normal to H_c, with respect to g; U^* is a constant unit normal to H_c^*, with respect to g^*; l is positive, and hyperbolic on g-geodesics; and l^* is affine on g^*-geodesics. Moreover, U and U^* are tangent to the same $(k+1)$-plane P through H_c^*. By definition of variation field, the following equation holds on $H_c = H_c^* \cap D^n$:

$$l^* = lU \cdot U^*. \tag{10}$$

Let q be the center and r_1 the radius of $H_c^* \cap D^n$, and r_2 the radius of $P \cap D^n$. By (10) and hyperbolicity of l, if p is a fixed point of H_c then

$$l^*(p) = r_2^{-1}\left(ar_1^2 + br_1 D^*(q,p)\right) \tag{11}$$

where $a = l(q)$, $b = Y(p)l$ and $Y(p)$ is the g-unit vector at q tangent to the g-ray γ from q toward p. Since l^* is affine, formula (11) in fact holds at any $p \in H_c^*$. Therefore we may choose p so that $p \in F_c$ and $D^*(q,p)$ realizes distance from q to F_c. Then $D^*(q,p) = -ab^{-1}r_1$ since $l^*(p) = 0$; and b (< 0) is minimized since $D^*(q,p)$ is minimized. By definition of b, it follows that the ray γ from q toward p is tangent to $-(\text{grad}_g l)_q$.

Now, $\sigma(c)$ is the striction point of $\{H_c\}$, that is, the minimum point of $\bar{l}$. Since l is hyperbolic, $\sigma(c)$ clearly lies on this same ray γ; and since $(l \circ \gamma)(s) = a \cosh s + b \sinh s$, then

$$D(q, \sigma(c)) = \tanh^{-1}(-a^{-1}b) \quad \text{and} \quad D^*(q, \sigma(c)) = -a^{-1}br_1.$$

Therefore p is the inversion of $\sigma(c)$ through $H_c^* \cap S^{n-1}$. This completes the proof of Theorem 7.

Note that it is a consequence of Theorem 7 that σ is smooth in $\overline{H}^n$. The following corollary gives certain properties which σ must possess at H^n_∞. Theorem 8 will show that these are in a sense exhaustive. (In the rest of this section, we use the following convention, based on the canonical identification with $\mathbf{R}^n$ of each tangent space to $\mathbf{R}^n$. If β denotes a Euclidean curve, $\beta'(c)$, $\beta''(c)$, and $\beta^{(3)}(c)$ denote elements of the tangent space at $\beta(c)$, unless these symbols occur in an algebraic expression, in which case they denote elements of $\mathbf{R}^n$.)

Corollary 2. *Let $\{H_c\}_{c\in I}$ be a family of hyperbolic k-planes which is parallel in $\overline{H}^n$ along a smooth curve σ: $I \to \overline{H}^n$, and suppose $\sigma(c) \in H^n_\infty$ for some c. Then $\sigma'(c)$ is tangent to $H_{c\infty}$, and $\sigma''(c)$ is tangent to $\overline{H}_c$.*

Proof. Note that we allow the possibility that $\{H_c\}$ is not regular at c, in which case the variation fields of both $\{H_c\}$ and $\{H_c^*\}$ vanish. By Theorem 7 (or trivially, in the nonregular case), $\sigma(c)$ lies in the focal set F_c of $\{H_c^*\}$, and so the variation field for $\{H_c^*\}$ vanishes at $\sigma(c)$. It follows from the definition of variation field and the fact that $\sigma(d) \in H_d^*$ for all $d \in I$ that $\sigma'(c)$ is tangent to H_c^*. Since σ is not transverse to S^{n-1}, $\sigma'(c)$ is tangent to $H_{c\infty} = H_c^* \cap S^{n-1}$.

Denote by $\sigma^*(d)$ the inversion of $\sigma(d)$ through $H_d^* \cap S^{n-1}$. By Theorem 7, $\sigma^*(d) \in F_d$ for all $d \in I$, hence $\sigma^{*\prime}(d)$ is tangent to H_d^*. In particular, since $\sigma^*(c) = \sigma(c)$ and σ^* is not transverse to S^{n-1}, $\sigma^{*\prime}(c)$ is tangent to $H_c^* \cap S^{n-1}$, and thus to F_c (since either $F_c = H_c^*$ or F_c is the hyperplane of H_c^* tangent to S^{n-1} at $\sigma(c)$). Setting $\zeta = \sigma^* + \sigma^{*\prime}$, we have $\zeta(d) \in H_d^*$ for all d, and $\zeta(c) \in F_c$. Therefore $\zeta'(c)$ is tangent to H_c^*. But $\sigma^{*\prime\prime}(c) = \zeta'(c) - \sigma^{*\prime}(c)$, so $\sigma^{*\prime\prime}(c)$ is also tangent to H_c^*. Moreover, it may be verified by direct computation that because σ and σ^* are inverse curves and $\sigma^{*\prime\prime}(c)$ is tangent to H_c^*, then $\sigma''(c)$ is tangent to H_c^*. Corollary 2 follows.

We are interested in regular families, since they always generate submanifolds. The following condition will be useful:

Corollary 3. *A parallel family $\{H_c\}$ along a regular curve σ: $I \to \overline{H}^n$ is* *regular if and only if* $\sigma'(c)$ *is transverse to* H_c *whenever* $\sigma(c) \in H_c$ *and* $\sigma^{(3)}(c)$ *is transverse to* $\overline{H}_c$ *whenever* $\sigma(c) \in H_{c\infty}$.

PROOF. (Note that by Theorem 2, for a parallel family regularity is equivalent to having no focal points.) When σ lies in H^n, the proof is the same as for Theorem 4.1. If $\sigma(c) \in H^n_\infty = S^{n-1}$, then $\sigma \cdot \sigma'' < 0$ at c because σ is regular; therefore $\sigma(c) + k\sigma''(c)$ lies in D^n for some $k > 0$, and hence in $H_c = H_c^* \cap D^n$ by Corollary 3.2. Now consider the curve $\zeta = \sigma + k\sigma''$. If $\sigma^{(3)}(c)$ is transverse to H_c^*, then so is $\zeta'(c)$, and it follows that $\{H_c\}$ is regular at c. If $\sigma^{(3)}(c)$ is tangent to H_c^*, then $\{H_c\}$ focuses at $\zeta(c)$, and so $\{H_c\}$ is not regular at c. Thus Corollary 3 is proved.

A parallelizing curve σ in $\overline{H}^n$ of a regular family of k-planes, while always regular where σ lies in H^n, need not be regular at H^n_∞. The regularity hypothesis on σ is necessary for the two uniqueness results which follow. Note that for a regular curve in D^n, the osculating 2-planes with respect to g and g^* agree, when they exist. Furthermore, for a regular curve in $\overline{D}^n$ and lying on S^{n-1} at c, the osculating 2-plane at c with respect to g^* exists and enters D^n. Therefore one may speak without ambiguity of the *osculating hyperbolic* 2-*plane* at c of any regular curve σ in $\overline{H}^n$ such that $\sigma(c) \in H^n_\infty$. For any parallel family along σ, the k-plane at c must contain this 2-plane, by Corollary 2.

THEOREM 8. *Let* $\sigma\colon I \to H^n_\infty$ *be a regular curve, where* $0 \in I$, *and* H_0 *be any hyperbolic* k-*plane containing the osculating hyperbolic* 2-*plane of* σ *at* $\sigma(0)$. *Then* H_0 *extends uniquely to a parallel family* $\{H_c\}_{c\in I}$ *along* σ.

PROOF. We reduce to a linear problem, that of parallel translation in the Euclidean normal bundle of σ (regarded as a curve lying on S^{n-1}). If $\{H_c\}$ is parallel along σ, then the corresponding Euclidean family $\{H_c^*\}$ has the following properties (by Corollary 2 and Theorem 7):

(i) H_c^* contains the osculating 2-plane of σ at $\sigma(c)$, and

(ii) the focal set F_c is either H_c^* or the hyperplane of H_c^* tangent to S^{n-1} at $\sigma(c)$.

Conversely, if $\{H_c^*\}$ is a Euclidean family satisfying (i) and (ii), then each H_c^* enters D^n by (i), and we claim that the corresponding hyperbolic family $\{H_c\}$ is parallel along σ. Where the families are not regular, constancy of the hyperbolic variation field on horospheres is obvious. Elsewhere, since each F_c is a $(k-1)$-plane not entering D^n, $\{H_c\}$ generates a submanifold of curvature -1 in H^n; thus by Theorems 2 and 7, $\{H_c\}$ is parallel along σ. Therefore it suffices, for the theorem, to demonstrate existence and uniqueness of a family $\{H_c^*\}$ of Euclidean k-planes satisfying (i) and (ii), when H_0^* is given.

Suppose such a family $\{H_c^*\}$ exists. Let $\mathcal{V}$ be the Euclidean vector bundle over σ whose $(k-2)$-dimensional fiber at each c is tangent to $H_c^* \cap S^{n-1}$

and orthogonal to $\sigma'(c)$. Then $H_c{}^*$ is tangent to $[\mathcal{V}, \sigma', \sigma'']_c$ (where bracket denotes span). Let v be a smooth section of $\mathcal{V}$. Setting $\zeta = \sigma + v$, we have $\zeta(c) \in F_c$ for all c, hence $\zeta'(c)$ tangent to $H_c{}^*$. Therefore $v'(c) \in [\mathcal{V}, \sigma', \sigma'']_c$. But $v' \cdot \sigma = 0$ since $v \cdot \sigma = v \cdot \sigma' = 0$, and so $v'(c) \in [\mathcal{V}, \sigma']_c$. Therefore $\mathcal{V}$ has parallel fiber in the Euclidean normal bundle of σ. Since $\mathcal{V}$ is uniquely determined by $H_0{}^*$ and σ, so is $\{H_c{}^*\}$.

On the other hand, for given $H_0{}^*$ let $\mathcal{V}_0$ consist of the vectors at $\sigma(0)$ tangent to $H_0{}^* \cap S^{n-1}$ and orthogonal to σ, and $\mathcal{V}$ be the parallel extension of $\mathcal{V}_0$ in the Euclidean normal bundle of σ. Then $\mathcal{V}$ is always tangent to S^{n-1}. Letting $H_c{}^*$ be the k-plane through $\sigma(c)$ tangent to $[\mathcal{V}, \sigma', \sigma'']_c$, we have a family satisfying (i). For (ii), it suffices to verify that $[\mathcal{V}, \sigma']_c$ is always tangent to the focal set F_c of $\{H_c{}^*\}$. But for any curve of the form $\zeta = \sigma + v + f\sigma'$ with v a section of $\mathcal{V}$, then $\zeta'(c) \in [\mathcal{V}, \sigma', \sigma'']_c$ and so $\zeta'(c)$ is tangent to $H_c{}^*$. Therefore $\zeta(c) \in F_c$, as required. Thus we have constructed a family $\{H_c{}^*\}$ satisfying (i) and (ii).

COROLLARY 4. *Let σ: $I \to \overline{H}^n$ be a regular curve, where $0 \in I$, and H_0 be a hyperbolic k-plane at $\sigma(0)$. If there exists a parallel family along σ extending H_0, then that family is unique.*

PROOF. Note that if two hyperbolic k-planes intersect, one may speak of the hyperbolic angle between them; this angle is measured in the $(n-r)$-plane which is g-orthogonal to the intersection r-plane P at any $p \in P$, and is independent of the choice of p. We shall show that any two parallel families $\{H_{1c}\}$ and $\{H_{2c}\}$ of k-planes along a given regular curve σ: $I \to \overline{H}^n$ have constant intersection dimension, and form a constant angle.

Note that $H_{1c} \cap H_{2c} \neq \varnothing$. Indeed, if $\sigma(c) \in H^n$ then $\sigma(c)$ lies in this intersection, and if $\sigma(c) \in H^n_\infty$ then this intersection contains a 2-plane at least. Let I_0 be a maximal subinterval of I on which the intersection dimension takes its minimum for I. Then I_0 is open in I. The angle between the two families is smooth on I_0, and certainly is constant on any subinterval on which σ lies entirely in H^n. Furthermore, it follows from the proof of Theorem 8 that the angle is constant on any subinterval J of I_0 on which σ lies entirely in H^n_∞: In the notation of that proof, suppose H_{ic} corresponds to $[\mathcal{V}_i, \sigma', \sigma'']_c$, for each $c \in J$. It may be verified that the hyperbolic angle between H_{1c} and H_{2c} is equal to the Euclidean angle between $[\mathcal{V}_1, \sigma']_c$ and $[\mathcal{V}_2, \sigma']_c$, since $[\mathcal{V}_i, \sigma']_c$ is tangent to $H_{ic}^* \cap S^{n-1}$ at $\sigma(c)$. In turn, the Euclidean angle between the $[\mathcal{V}_i, \sigma']$ equals that between the $\mathcal{V}_i$ since the $\mathcal{V}_i$ are orthogonal to σ'. This angle is constant on J because the $\mathcal{V}_i$ are parallel in the Euclidean normal bundle of $\sigma|J$.

Therefore the hyperbolic angle between the two families has vanishing derivative on a dense subset of I_0, hence is constant on I_0. It follows that I_0 is closed in I, and $I_0 = I$.

REMARK 6. Given a regular curve σ: $I \to \overline{H}^n$, and an initial k-plane H_0, there may be no parallel family on σ extending H_0. For example, if $\sigma(0) \in H^n$, the the parallel translate of H_0 along σ as σ approaches H^n_∞ may have more than one limit k-plane, or none at all.

5. **Examples.** By way of illustration, we examine families of hyperbolic k-planes which have parallelizing curves in $\overline{H}^n$ of special type. Our conventions are the same as in §4.

EXAMPLE 2. A family $\{H_c\}_{c \in I}$ without focal points is parallel along an orthogonal trajectory σ: $I \to H^n$ if and only if the corresponding Euclidean k-planes H_c^* all pass through a fixed $(k - 1)$-plane not meeting $\overline{D}^n$. This may be seen as follows.

Note that if σ is both parallelizing and orthogonal, then σ lies in a hyperbolic $(n - k)$-plane orthogonal to the H_c. To see this, observe that the family $\{H_c^\perp\}$ of hyperbolic $(n - k)$-planes, where $H_c^\perp \cap H_c = \sigma(c)$ and $H_c^\perp$ is orthogonal to H_c, is parallel along and tangent to σ. Thus the variation field at c of $\{H_c^\perp\}$ vanishes at $\sigma(c)$, and, by Theorem 2, vanishes identically. Therefore the $H_c^\perp$ coincide, in an $(n - k)$-plane containing σ. Now, it can be shown that a hyperbolic k-plane is orthogonal to a given hyperbolic $(n - k)$-plane H if and only if the corresponding Euclidean k-plane lies in the pencil through a certain Euclidean $(k - 1)$-plane, not meeting $\overline{D}^n$, determined by H. Thus, since the H_c are all orthogonal to a fixed $(n - k)$-plane, it follows that the H_c^* all pass through a fixed $(k - 1)$-plane.

The converse follows from Theorem 7 and the fact that, if $\{H^*\}$ now denotes the collection of Euclidean k-planes passing through a fixed $(k - 1)$-plane F and entering D^n, the dual points of F in all the H^* form a hyperbolic $(n - k)$-plane which is g-orthogonal to each $H = H^* \cap D^n$.

EXAMPLE 3. A family $\{H_c\}$ has constant parallelizing curve in H^n_∞ if and only if the H_c^* all pass through a fixed $(k - 1)$-plane tangent to S^{n-1}. Here it suffices to verify that if the parallelizing curve σ is constant at H^n_∞, then the focal family $\{F_c\}$ of $\{H_c^*\}$ is constant. If $\{F_c\}$ is not constant, then at some c the variation field of $\{F_c\}$ does not vanish identically on F_c. This field is defined on the hyperplane F_c of H_c^* and is tangent to H_c^*. Since all $F_{c+\delta}$ lie in the tangent plane to S^{n-1} at the image point of σ, it follows that H_c^* is tangent to S^{n-1}, which is impossible since H_c^* enters D^n.

EXAMPLE 4. The osculating Euclidean 2-planes of a regular curve σ lying on S^{n-1} and having nowhere vanishing torsion (second order curvature function) correspond to a regular family of hyperbolic 2-planes which is parallel along σ. Indeed, parallelism along σ follows from the proof of Theorem 8. Regularity is a consequence of Corollary 3 and the torsion assumption. Conversely, all regular hyperbolic 2-plane families which are parallel along a regular curve in H^n_∞ are obtained in this way, by Corollaries 2 and 3.

Now suppose $n = 3$, and consider all foliations of H^3 by 2-plane families whose parallelizing curves are regular curves in H^3_∞. These foliating families correspond precisely to the families of osculating 2-planes of regular Euclidean curves lying on S^2, for which torsion is nonvanishing and curvature (denoted here by κ^*) becomes infinite at both endpoints. For example, a loxodrome is such a curve. It is only necessary to verify that for a unit-speed curve σ on S^2, the osculating planes approach tangency to S^2 at an endpoint if and only if curvature becomes infinite. This follows by differentiating $\sigma \cdot \sigma' = 0$, to obtain $|\sigma \cdot \sigma''/\|\sigma''\|| = 1/\kappa^*$.

EXAMPLE 5. Let $\{H_c^*\}$ be any regular family of $(n-1)$-planes in E^n such that each member intersects D^n. The equation of H_c^* is $y \cdot N(c) - r(c) = 0$, where N and r denote (smoothly varying) unit normal and distance from the origin respectively. Then the focal $(n-2)$-plane F_c in H_c^* satisfies $y \cdot N'(c) - r'(c) = 0$. (Indeed, $y \in F_c$ if and only if there is a curve β satisfying $\beta(c) = y$, $\beta(c + \delta) \in H^*_{c+\delta}$, and $\beta'(c) \cdot N(c) = 0$; but the last two conditions imply $\beta(c) \cdot N'(c) - r'(c) = 0$.) If we assume that $\{F_c\}$ never enters D^n, then $\{H_c^*\}$ determines a regular hyperbolic family which is parallel along some curve σ in $\overline{H}^n$. By Theorem 7, the equations for $\{F_c\}$ yield a formula for σ:

$$\sigma = rN + (1 - r^2)(r')^{-1}N'. \tag{12}$$

Here r' never vanishes because $\{F_c\}$ never enters D^n.

This formula may be used to construct examples, such as the following one for $n = 2$. The curve $\sigma^*(c) = (1 + c^2, c^3)$ lies outside S^1 and meets S^1 at $c = 0$, in a cusp tangent to the x-axis. The tangent lines H_c^* to σ^* form a regular Euclidean family (even though σ^* is not regular). The focal sets of $\{H_c^*\}$ are given by $F_c = \sigma^*(c)$, and hence lie outside D^2. Thus for a certain symmetric interval I about 0, the tangent lines to $\sigma^*|I$ determine a family $\{H_c\}$ of hyperbolic lines foliating H^2. The parallelizing curve σ may be computed from (12) to be

$$\sigma(c) = (3c^4 + 2c^2 + 2)^{-1}(c^6 + 3c^4 + 2, -c^5 - 4c^3).$$

Note that $\sigma'(0) = 0$, as required by Corollary 3, and that σ meets S^1 at $c = 0$, in a cusp tangent to the x-axis.

REMARK 7. Finally, we mention one further possible direction for investigation: With regard to §4, one might ask precisely which curves $\sigma: I \to \overline{H}^n$ (taking values in both H^n and H^n_∞) allow the existence of parallel families.

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